The Precessing Top

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1. Introduction

When a symmetric top is set spinning with angular velocity ω_c about its own axis, it traces out a circle about a vertical direction with angular velocity ω_z, as shown in Fig. 1.

In most textbooks [1, 2, 3] the *precessional* angular velocity ω_z is calculated under the assumption that it is much smaller than the *spin* angular velocity ω_c. This makes for a simplification of the calculation by considering the total angular momentum as due to the spin motion only, and thus directed along the symmetry axis.

In this case, the angular momentum along the symmetry axis is $L = I\omega_c$, where I is the moment of inertia about this same axis. The tip of this vector describes a circle of radius $L\sin\theta$ around the vertical direction. The torque of gravity is $mgl\sin\theta$, where m is the mass of the top and l is the distance of its center of mass C to O. In a time interval Δt the torque changes the angular momentum by $L\sin\theta\omega_z\Delta t$ in a direction tangent to the circle. Equating torque to time change in angular momentum yields the usual formula $\omega_z = mgl/I\omega_c$ for slow precession [2].

In this note I drop the simplifying assumption that the total angular momentum is solely along the symmetry axis in order to obtain the following general expression for ω_z,

$$\omega_z = \frac{I\omega_c \pm \sqrt{I^2\omega_c^2 + 4(I - I_n)mgl\cos\theta}}{2(I_n - I)\cos\theta}\ , \tag{0.1}$$

where I_n is the moment of inertia of the top about any axis normal to the symmetry axis at O. There are two values for ω_z if $I^2\omega_c^2 + 4(I - I_n)mgl\cos\theta > 0$. This condition is always satisfied when $I > I_n$. When $I < I_n$ it determines the minimum velocity $\omega_c = (2/I)\sqrt{(I_n - I)mgl\cos\theta}$ for a steady precession to occur.

Formula (0.1) is most useful for practical purposes when ω_c is very large, in which case the approximation $\sqrt{1+x} = 1 + x/2$ for the square root in (0.1) when $x = 4(I - I_n)mgl\cos\theta/I^2\omega_c^2$ is very

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small gives

$$\omega_z = \frac{I\omega_c}{2(I_n - I)\cos\theta} \pm \left(\frac{I\omega_c}{2(I_n - I)\cos\theta} - \frac{mgl}{I\omega_c} \right). \tag{0.2}$$

This approximation for ω_z yields not only the previous formula for slow precession $\omega_z^{(-)} = mgl/I\omega_c$, but also the formula for *fast* precession $\omega_z^{(+)} = I\omega_c/(I_n - I)\cos\theta$.

In this last case, the sense of precession depends on whether $I < I_n$ or $I > I_n$. For the top shown in Fig. 1, and for most ordinarily shaped child's top, $I_n > I$.

Which precession occurs depends on how the top is set in motion. Once it is started at an angle θ to the vertical with angular velocity ω_c and either of the above values for ω_z, it will continue to precess steadily.

2. Analysis

The analysis that leads to (0.1) is elementary and amounts to taking into account the angular momentum about the vertical direction. The angular velocity ω_z has components $\omega_z \cos\theta$ along the symmetry axis, and $\omega_z \sin\theta$ along an axis normal to it, as shown in Fig. 2. Thus the angular momentum about the vertical direction can be decomposed into a component $I\omega_z \cos\theta$ along the symmetry axis, and a component $I_n\omega_z \sin\theta$ normal to the symmetry axis. Since these two components also describe a circle around the vertical direction, the previous simplified analysis can be applied separately here for the components along and normal the symmetry axis. Thus the correction for the change in total angular momentum in a time interval Δt has now two steps.

a. The change in the component of the angular momentum along the symmetry axis is what we had before, with the addition of the extra term $I\omega_z \cos\theta$ due to the precession motion. It becomes $(L + I\omega_z \cos\theta)\sin\theta\omega_z\Delta t$.

b. The component of the angular momentum normal to the symmetry axis is $L_n = I_n\omega_z \sin\theta$. The tip of this vector describes a circle of radius $L_n \cos\theta$ around the vertical direction. The torque of gravity changes this component in a direction tangent to this circle. Thus the change in the normal component is given by $L_n \cos\theta\omega_z\Delta t$.

Since the changes in *(a)* and *(b)* are opposite in direction, equating torque to time change in angular momentum now leads to the equation $(I - I_n)\omega_z^2 \cos\theta + (I\omega_c)\omega_z - mgl = 0$ for ω_z, whose solutions are given by (0.1).

References

[1] D. Halliday, R. Resnick, *Fundamentals of Physics*, 3rd. ed., John Wiley, NY (1988) p. 272

[2] R. A. Serway, *Physics for Scientists and Engineers*, Saunders College Publishing, PA (1996) p. 321

[3] R. Skinner, *Mechanics*, Blaisdell Publushing Company, MA (1969) p. 475

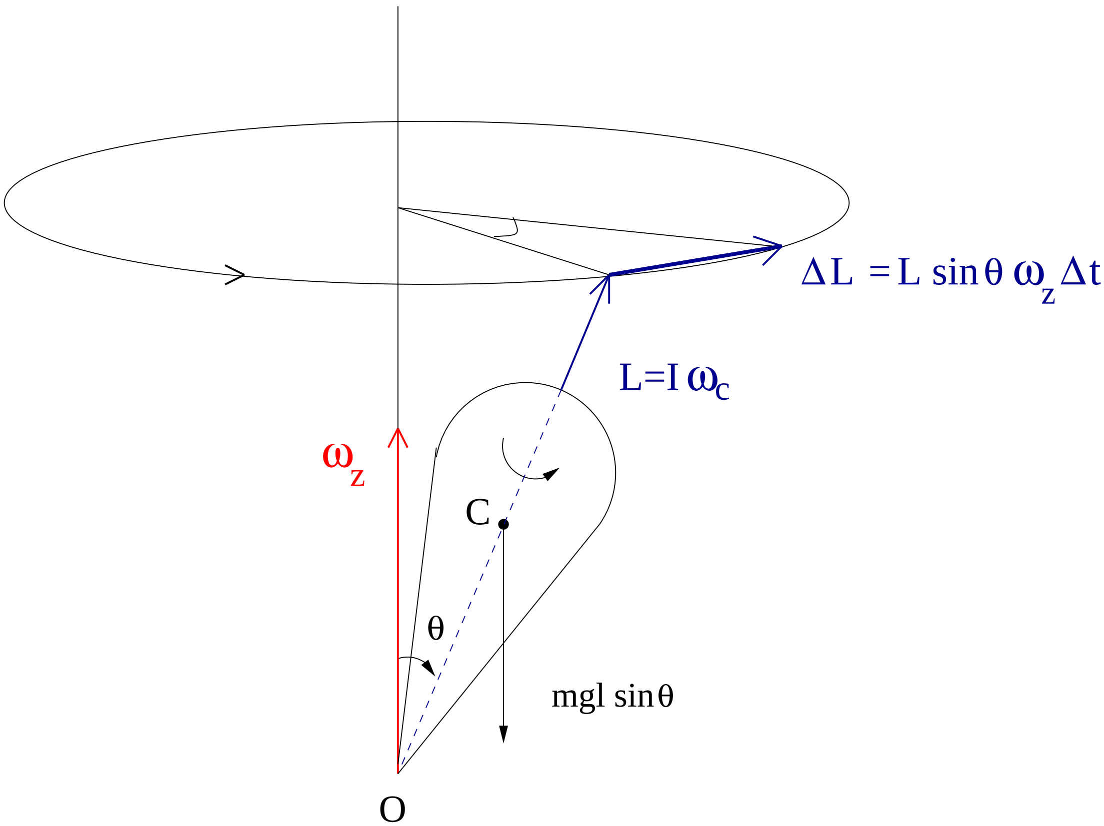

Figure 1: In the simplified analysis the total angular momentum of the top $L = I\omega_c$ is due solely to the rotation around its axis of symmetry.

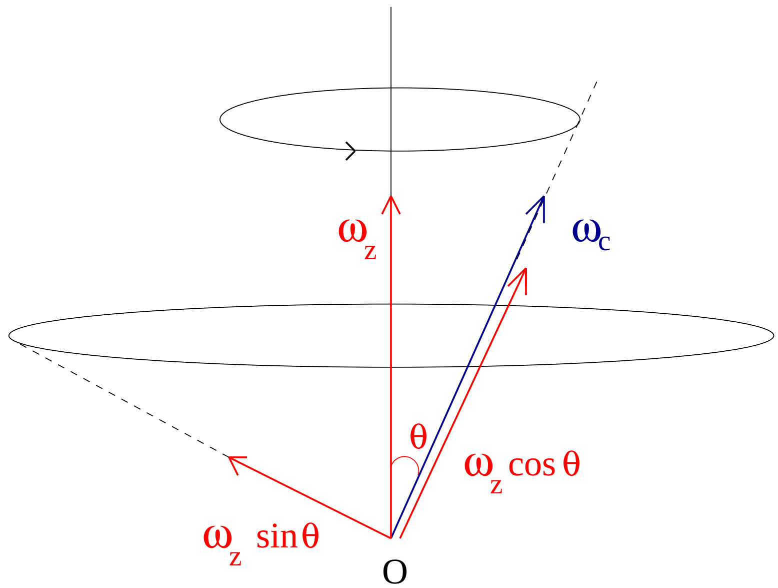

Figure 2: In the general analysis, the angular momentum of the rotational motion around the vertical axis is taken into account.